In a nutshell: Error analysis of single-step methods

Given the initial-value problem (IVP)

$$y^{(1)}(t) = f(t, y(t))$$
$$y(t_0) = y_0$$

we would like to approximate the solution $y(t)$ on the interval $[t_0, t_f]$. We know that Euler's, Heun's and the 4th-order Runge-Kutta methods are order h, h^2 and h^4, respectively, and we can approximate the error starting with N steps as follows:

1. Set $h \leftarrow \frac{t_f - t_0}{N}$.
2. Use the method to approximate N values $y_0, y_1, y_2, \ldots, y_N$ using h.
3. Use the same method to approximate $2N$ values $y_0, z_1, z_2, \ldots, z_{2N}$ using ½h.
4. The error of z_{2N} as an estimate of $y(t_f)$ is $z_{2N} - y_N$, $\frac{z_{2N} - y_N}{3}$ and $\frac{z_{2N} - y_N}{15}$ for these three methods, respectively.
5. If the error is sufficiently small, then the best approximations are $y(t_k) \approx 2z_{2k} - y_k$, $y(t_k) \approx \frac{4z_{2k} - y_k}{3}$ and $y(t_k) \approx \frac{16z_{2k} - y_k}{15}$ with $k = 1, \ldots, n$ for these three methods, respectively;

   otherwise, halve h and return to Step 2, now using $2N$ and $4N$ points.